Reflection and transmission of polarized light by optically thick weakly absorbing random media

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Simple analytical relations for reflection and transmission matrices of plane-parallel layers of random media with discrete particles are presented. They can be used for rapid estimation of intensity and polarization characteristics of reflected and transmitted light beams under arbitrary illumination of a layer. The accuracy of the analytical formulas obtained increases with the optical thickness τ of a layer. Thus equations are applicable only at large values of $\tau\,(\tau > 5)$. Another limitation is due to the probability of photon absorption β, which should be rather low ($\beta < 0.05$–0.1, depending on the optical characteristics in question). © 2001 Optical Society of America

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1. INTRODUCTION

Reflection and transmission matrices of various disperse media are usually found by solving the integro-differential vector radiative transfer equation.[1] The radiative transfer equation can be applied to a broad range of physical problems, including neutron transport, atmospheric, oceanic, and tissue optics. It is solved numerically in most of cases. However, analytical solutions of this equation have also been derived for many particular situations, e.g., for optically thin and optically thick layers.[1] Formulas for thin layers are especially simple and have already been used in many applications.[2] The use of asymptotic analytical equations for optically thick layers is not so common, because asymptotic formulas are expressed in terms of solutions of integral equations.[1] One can argue that there is no point in using asymptotic solutions if they require complex numerical codes for calculation of the auxiliary functions and parameters involved. Perhaps it is better to start from the radiative equation from the very beginning in this case.

However, simple relations between global optical characteristics (e.g., the reflection function and the degree of polarization) of different disperse media and their local optical characteristics (e.g., the absorption coefficient and the phase matrix) are especially important for the solution of inverse problems. Thus it is the aim of this paper to derive such equations for weakly absorbing plane-parallel optically thick light-scattering layers. The assumption of low level of light absorption in a medium is used to simplify the general asymptotic solutions, which have already been known for a long time.[3,4] Note that weakly absorbing optically thick layers occur frequently in nature. Snow and ice fields, fog and clouds, blood, paper, paints and many other types of natural media belong to this important subclass of disperse media.

2. ASYMPTOTIC THEORY

Let us start from asymptotic formulas of the vector radiative transfer theory.[1] Corresponding equations can be written in terms of reflection $\hat{R}$ and transmission $\hat{T}$ matrices. These matrices relate Stokes vectors of transmitted $\mathbf{S}_t$ and reflected $\mathbf{S}_r$ light beams to the Stokes vector of the incident light beam $\mathbf{F}$:

$$\mathbf{S}_t(\mu, \phi) = \hat{T}(\mu, \mu_0, \phi)\mathbf{F},$$

$$\mathbf{S}_r(\mu, \phi) = \hat{R}(\mu, \mu_0, \phi)\mathbf{F}$$

Here $\mu = \cos\theta$; $\mu_0 = \cos\theta_0$; θ_0 and θ are incidence and observation angles, respectively; ϕ is the relative azimuth of incident and reflected or transmitted light beams; and we assume that light comes from just one direction specified by the cosine of the incidence angle μ_0 and the azimuth ϕ_0. Note that the vector $\mathbf{F}$ is normalized in such a way that the first element of $\pi\mathbf{F}$ is the net flux of a light beam per unit area of a disperse medium.

Thus we will use the following asymptotic expressions[1–5] for the reflection matrix $\hat{R}(\mu, \mu_0, \phi)$ and the transmission matrix $\hat{T}(\mu, \mu_0)$ of a homogeneous plane-parallel turbid layer of large optical thickness $\tau = \sigma_{\text{ext}} L$ (σ_{ext} is the extinction coefficient of the disperse slab in question, and L is the geometrical thickness of the slab):

$$\hat{R}(\mu, \mu_0, \phi) = \hat{R}_\infty(\mu, \mu_0, \phi) - f\,\hat{T}(\mu, \mu_0). \tag{1}$$

$$\hat{T}(\mu, \mu_0) = t\mathbf{K}(\mu)\mathbf{K}^T(\mu_0), \tag{2}$$

where

$$t = \frac{m \exp(-k\tau)}{1 - f^2}. \tag{3}$$

It should be pointed out that matrices $\hat{R}(\mu, \mu_0, \phi)$ and $\hat{T}(\mu, \mu_0)$ can be used to calculate the Stokes vectors of reflected and transmitted light beams under arbitrary illumination of a scattering layer.[1] Equations (1) and (2) follow from the exact vector radiative transfer equation as $\tau \to \infty$. Thus they are exact formulas in the asymptotic limit. The details of the derivation of these important equations were presented for the first time by Domke.[4]

Note that the vector $\mathbf{K}^T(\mu_0)$ is transpose to the vector $\mathbf{K}(\mu_0)$. Constants f and m and the vector $\mathbf{K}(\mu)$ can be found from following relations[4,5]:

$$f = s \exp(-k\tau), \tag{4}$$

$$m = 2\int_0^1 \mathrm{d}\mu\mu[\mathbf{P}^T(\mu)\mathbf{P}(\mu) - \mathbf{P}^T(-\mu)\mathbf{P}(-\mu)], \tag{5}$$

$$\mathbf{K}(\mu) = m^{-1}\left[\mathbf{P}(\mu) - 2\int_0^1 d\xi\xi\hat{R}_\infty(\mu, \xi)\mathbf{P}(-\xi)\right], \tag{6}$$

where $\mathbf{P}^T$ means the transpose vector to the vector $\mathbf{P}$,

$$s = 2\int_0^1 \mathrm{d}\mu\mu\mathbf{K}^T(\mu)\mathbf{P}(-\mu), \tag{7}$$

and

$$\hat{R}_\infty(\mu, \xi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\hat{R}_\infty(\mu, \xi, \phi) \tag{8}$$

is the azimuthally averaged reflection matrix of the semi-infinite medium with the same local optical characteristics as those for the finite layer under study. Note that the escape vector $\mathbf{K}(\mu)$ also occurs in the so-called Milne problem. It describes the angular dependence of the intensity and polarization characteristics of light emerging from semi-infinite turbid layers with sources of radiation located deep inside the medium. The constant k is the smallest discrete eigenvalue that is real and nondegenerate and obeys the following equation:

$$(1 - k\mu)\mathbf{P}(\mu) = \frac{\omega_0}{2}\int_{-1}^1 \mathrm{d}\xi\hat{Z}(\mu, \xi)\mathbf{P}(\xi), \tag{9}$$

which is called the vector characteristic equation of the radiative transfer theory. Here μ is the cosine of the observation angle, ω_0 is the single scattering albedo, which is equal to the ratio of the scattering and extinction coefficients, $\hat{Z}(\mu, \xi)$ is the azimuth-averaged phase matrix, and $\mathbf{P}(\xi)$ is the corresponding eigenvector.

Equations (1) and (2) are very general and can be applied to disperse media with arbitrary local optical characteristics, assuming that the optical thickness of the light-scattering medium in question is large. However, their application to the solution of practical problems is not easy, because it is necessary to solve integral equation (9) and perform integrations (5)–(8). The reflection matrix $\hat{R}_\infty(\mu, \xi, \phi)$ obeys the integral equation as well.[5]

The problem of finding matrices $\hat{R}(\mu, \mu_0, \phi)$ and $\hat{T}(\mu, \mu_0)$ in Eqs. (1) and (2) can be simplified for a broad class of so-called weakly absorbing media, which are characterized by a small value of the probability of photon absorption $\beta = 1 - \omega_0$. One can obtain,[1] at small values of β,

$$k = [3(1 - g)(1 - \omega_0)]^{1/2}, \tag{10}$$

$$m = \frac{8k}{3(1 - g)}, \tag{11}$$

$$s = 1 - \frac{4k\alpha}{3(1 - g)}, \tag{12}$$

$$\hat{R}_\infty(\mu, \mu_0, \phi) = \hat{R}_\infty^0(\mu, \mu_0, \phi) - \frac{4k}{3(1 - g)}\bar{\mathbf{K}}_0(\mu)\mathbf{K}_0^T(\mu_0), \tag{13}$$

where $\mathbf{K}_0(\mu)$ and $\hat{R}_\infty(\mu, \mu_0, \phi)$ are the escape vector and the reflection matrix, respectively, of the nonabsorbing medium with the same phase matrix as the finite absorbing layer under study,

$$g = \frac{1}{2}\int_0^\pi p(\theta)\sin\theta\cos\theta\mathrm{d}\theta \tag{14}$$

is the asymmetry parameter of the phase function $p(\theta)$, and

$$\alpha = 3\int_0^1 \mathrm{d}\mu\mu^2 S(\mu), \tag{15}$$

where $S(\mu) = \mathbf{K}_0^T(\mu)\mathbf{e}_1$, $\mathbf{e}_1$ is the unit column vector. It should be pointed out that integral (15) is close to 1 for all types of phase matrices.[1,2] Thus we will neglect small differences $\Delta = 1 - \alpha$ in the discussion that follows. Note that the first moment of the function $S(\mu)$ does not depend on the phase matrix at all, because of the normalization condition[2]:

$$2\int_0^1 \mathrm{d}\mu\mu S(\mu) = 1. \tag{16}$$

Thus Eqs. (10)–(13) allow us to reduce the complexity of Eqs. (1) and (2). Only function $\mathbf{K}_0(\mu)$ and $\hat{R}_\infty^0(\mu, \mu_0, \phi)$ for a nonabsorbing medium should be found numerically in this simpler case. They depend on the phase matrix of the random medium in question. However, they do not depend on the single-scattering albedo ω_0 and the optical thickness τ. Another interesting feature of these functions is their weak dependence on the microstructure parameters (size, shape, and chemical composition of particles) of the medium under study in the broad range of angular parameters μ, μ_0, and ϕ. This is due to the randomization of both the polarization states and the propagation directions of photons in highly scattering semi-infinite layers irrespective of the local optical properties of the medium in which they propagate.

3. MODIFIED ASYMPTOTIC EQUATIONS

A. Main Formulas

Numerical calculations show that expansions (11)–(13) can be applied only for extremely small values of $\beta = 1 - \omega_0$ ($\beta \sim 10^{-4}$). This greatly limits the power of the asymptotic theory for real-life problems.

Various methods to overcome this problem have been used. The most straightforward approach calls for the derivation of higher-order terms in expansions (10)–(13). However, here we will use another method, which was

initially applied in scalar radiative transfer theory.[6] According to this alternative approach, expansions (11) and (12) are replaced by exponential functions,

$$m = 1 - \exp(-2y), \tag{17}$$

$$s = \exp(-y), \tag{18}$$

where

$$y = \frac{4k}{3(1-g)}, \tag{19}$$

and we assume that $\alpha = 1$. Equations (17) and (18) should be regarded as empirical relations; they cannot be derived from first principles. Formulas (17) and (18) transform to exact asymptotic results (11) and (12) as $k \to 0$, provided that $\alpha = 1$. These equations do allow us to calculate radiative and polarization characteristics for many types of absorbing light-scattering media with values of $\beta \leqslant 0.1$, as follows from the results presented below.

Thus Eqs. (1) and (2) can be transformed into the following forms:

$$\hat{R}(\mu, \mu_0, \phi) = \hat{R}_\infty(\mu, \mu_0, \phi) - \hat{T}(\mu, \mu_0)\exp(-y - k\tau), \tag{20}$$

$$\hat{T}(\mu, \mu_0) = t\mathbf{K}_0(\mu)\mathbf{K}_0^T(\mu_0), \tag{21}$$

where we have accounted for Eqs. (17) and (18) and the approximate equality

$$m\mathbf{K}(\mu)\mathbf{K}^T(\mu_0) \approx [1 - \exp(-2y)]\mathbf{K}_0(\mu)\mathbf{K}_0^T(\mu_0). \tag{22}$$

The value of

$$t = \frac{1 - \exp(-2y)}{\exp(k\tau) - \exp(-2y - k\tau)} \tag{23}$$

is the global transmittance, defined as[1]

$$t = 4\int_0^1 \mu \mathrm{d}\mu \int_0^1 \mu_0 \mathrm{d}\mu_0 T_{11}(\mu, \mu_0). \tag{24}$$

Equation (24) follows from Eq. (21), accounting for the integral relation,[1]

$$2\int_0^1 \mu \mathrm{d}\mu K_{01}(\mu)\mathrm{d}\mu = 1, \tag{25}$$

where $K_{01}(\mu)$ is the first component of the escape vector $\mathbf{K}_0(\mu)$.

Note that the global transmittance (24) could be transformed into the following simple form[6]:

$$t = \frac{sh y}{sh(x + y)}, \tag{26}$$

where

$$x = k\tau. \tag{27}$$

Equations (20) and (21) are much simpler than initial asymptotic formulas (1) and (2). They allow us to calculate the reflection and transmission matrices of optically thick weakly absorbing disperse media by simple means, if the solution of the problem for a semi-infinite medium with the same phase matrix as for the finite layer under consideration is available. This reduction of the general problem to the case of a semi-infinite medium is important for radiative transfer theory.

It follows from Eqs. (20) and (21), neglecting the polarization characteristics of a light beam under study, that

$$R(\mu, \mu_0, \phi) = R_\infty(\mu, \mu_0, \phi) - T(\mu, \mu_0)\exp(-y - k\tau), \tag{28}$$

$$T(\mu, \mu_0) = tK_0(\mu)K_0(\mu_0), \tag{29}$$

where $T \equiv T_{11}$, $R \equiv R_{11}$, $R_\infty^0 \equiv R_{\infty 11}^0$, $K_0 \equiv K_{01}$ are the first elements of the corresponding matrices. Equations (28) and (29) represent well-known results of scalar radiative transfer theory.[6,7] This confirms our calculations.

Note that formulas (20) and (21) can be simplified for nonabsorbing media ($k = 0$):

$$\hat{R}(\mu, \mu_0, \phi) = \hat{R}_\infty^0(\mu, \mu_0, \phi) - \hat{T}(\mu, \mu_0), \tag{30}$$

$$\hat{T}(\mu,\mu_0) = t\mathbf{K}_0(\mu)\mathbf{K}_0^T(\mu_0), \tag{31}$$

where [see Eq. (23) as $k \to 0$]

$$t = 1/[1 + \tfrac{3}{4}\tau(1 - g)]. \tag{32}$$

Let us apply Eqs. (30) and (31) to a particular problem, namely, to the derivation of a relation between the spherical albedo $r = 1 - t$ and the degree of polatization of reflected light $P(\mu_0)$ at the nadir viewing geometry ($\mu = 1$). We will assume that a scattering layer is illuminated on the top by a wide unidirectional unpolarized light beam. The value of $P(\mu_0)$ is given simply by $-R_{21}(\mu_0, 1)/R_{11}(\mu_0, 1)$ in this case. Thus it follows from Eqs. (30) and (31) that

$$P(\mu_0) = \frac{P_\infty(\mu_0)}{1 - u(\mu_0)(1 - r)}, \tag{33}$$

where

$$u(\mu_0) = \frac{K_{01}(1)K_{01}(\mu_0)}{R_{\infty 11}(\mu_0, 1)}, \qquad P_\infty(\mu_0) = -\frac{R_{\infty 21}(\mu_0, 1)}{R_{\infty 11}(\mu_0, 1)}, \tag{34}$$

and we have accounted for the equality[2] $K_{02}(1) = 0$. Our calculations show that the value of $u(\mu_0)$ is close to 1 for most of the observation angles, which implies inverse proportionality [see Eq. (33)] between the brightness of a turbid medium and the degree of polarization of reflected light.

This inverse proportionality between the spherical albedo r and the degree of polarization $P(\mu_0)$ was discovered experimentally by Umow[8] almost 100 years ago. Equation (33) can be considered a manifestation of this important law, which has important applications in planetary spectroscopy.[9]

B. Numerical Results

Let us apply the equations derived to the specific case of water clouds illuminated by solar light. Numerical calculations with Eq. (34) show that the function $u(\mu_0)$ varies from 1 to 1.36 with the incidence angle in this case. However, this variation is of secondary importance in cal-

culations of the degree of polarization because of small values of the difference $1 - r$ for weakly absorbing optically thick media [see Eq. (33)]. Thus one can neglect the angular dependence $u(\mu_0)$ and use the average value $u(\mu_0) = 1.18$ in the case under study. It follows that Eq. (33) can be written in the following simplified form:

$$P(\mu_0) = \frac{P_\infty(\mu_0)}{1 - 1.18t}, \tag{35}$$

where global transmittance t is given by Eq. (32). Let us study the accuracy of this simple equation. The accuracy of Eq. (35) can be seen from Fig. 1, where data obtained from this simple formula and results of the numerical solution of the vector radiative transfer equation for light-scattering media with water droplets by the doubling method[1] are presented. The value of $P_\infty(\mu_0)$ was obtained from the exact radiative transfer calculations for a semi-infinite layer. The particle size distribution $\psi(a)$ in a cloud layer was given by the following formula:

$$\psi(a) = Aa^\mu \exp\left(-\mu \frac{a}{a_0}\right),$$

where A is the normalization constant ($\int_0^\infty f(a)\mathrm{d}a = 1$), a is the radius of particles, $a_0 = 4\ \mu\mathrm{m}$, and $\mu = 6$. This distribution is often used in cloud optics. It is called the gamma size distribution. The optical thickness of a layer was equal to 8, 15, and 30. Calculations of the phase matrix were performed with Mie theory at a wavelength of 443 nm, where light absorption by water droplets can be neglected. The obtained value of the asymmetry parameter was equal to 0.8541 in the case under investigation.

It follows from Fig. 1 that Eq. (35) describes the degree of polarization of reflected light with high accuracy at $\tau \geqslant 8$. As a matter of fact, this equation can be applied to media with optical thickness as low as 5 (see Fig. 2).

Equations (33) and (34) can be easily generalized to account for the absorption of light in a medium: It follows from Eq. (20) for the value of $P = -R_{21}(\mu_0, 1)/R_{11}(\mu_0, 1)$ that

$$P(\mu_0) = -\frac{R_{\infty 21}(\mu_0, 1)}{R_{\infty 11}(\mu_0, 1) - K_{01}(\mu)K_{01}(\mu_0)t \exp(-x - y)}$$

or

$$P(\mu_0) = \frac{P_\infty^*(\mu_0)}{1 - u^*(\mu_0)t \exp(-x - y)}, \tag{36}$$

where the global transmittance t is given by Eq. (26) and

$$u^*(\mu_0) = \frac{K_{01}(1)K_{01}(\mu_0)}{R_{\infty 11}^*(\mu_0, 1)}. \tag{37}$$

Values of $P_\infty^*(\mu_0)$ and $R_\infty^*(\mu_0, 1)$ represent the degree of polarization and the reflection function of a semi-infinite weakly absorbing medium at the nadir viewing geometry. Note that Eq. (36) can be written in the following form:

$$P(\mu_0) = c(\mu_0, \tau)P_\infty^*(\mu_0), \tag{38}$$

where the value of

$$c(\mu_0, \tau) = \frac{1}{1 - u^*(\mu_0)t \exp(-x - y)} \tag{39}$$

can be interpreted as the enhancement factor, which is due to the finite depth of a turbid layer. Equation (38) can be used to estimate the change in the degree of polarization with the optical thickness of the medium in question.

It is evident that the value of $c \to 1$ as $\tau \to \infty$, as it should be. Also it follows from Eq. (38) that zeros of polarization curves for semi-infinite and optically thick finite layers coincide, which is supported by numerical calculations with the radiative transfer code (see Fig. 1). This is due to the fact that multiple light scattering fails

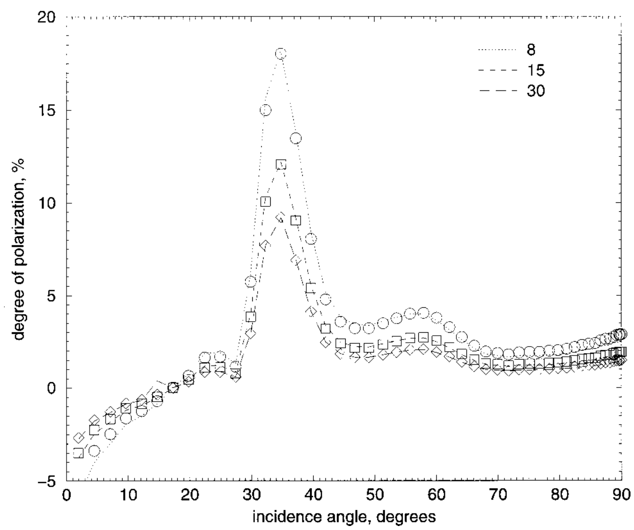

Fig. 1. Dependence of the degree of light polarization reflected from a water cloud with gamma particle size distribution at a wavelength of 443 nm on the incidence angle at nadir observation at $\tau = 8$, 15, and 30 according to numerical radiative transfer calculations (broken curves) and Eqs. (35) and (32) (symbols).

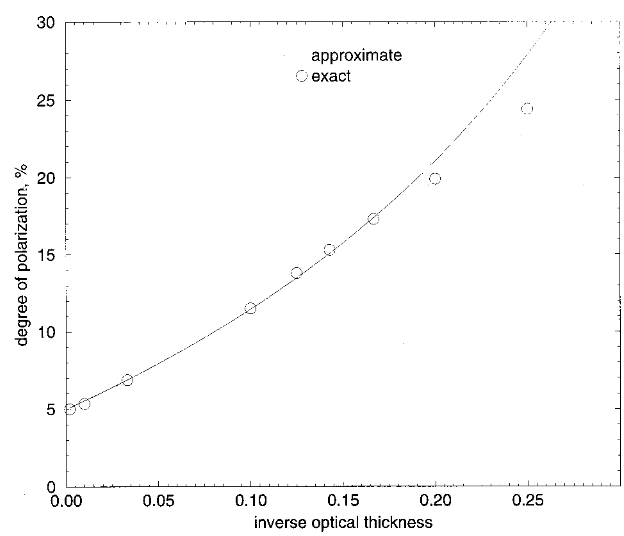

Fig. 2. Dependence of the degree of light polarization reflected from a water cloud with gamma particle size distribution at a wavelength of 443 nm on the inverse optical thickness at nadir observation and incidence angle equal to 37° according to approximate Eq. (35) (solid curve) and numerical radiative transfer calculations (symbols).

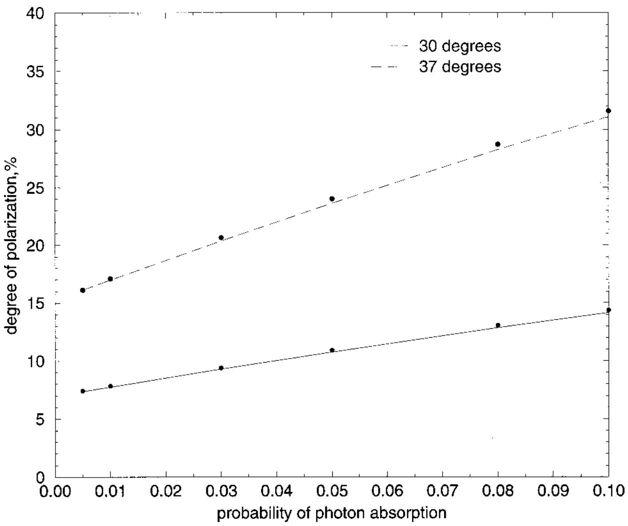

Fig. 3. Dependence of the degree of light polarization reflected from a water cloud on the probability of photon absorption at nadir observation and incidence angle equal to 30° and 37° according to numerical radiative transfer calculations (symbols) and Eq. (36) (lines) at optical thickness 7. The phase matrix coincides with the phase matrix used for calculations presented in Figs. 1 and 2.

to polarize incident unpolarized light. It only diminishes the polarization of singly scattered light. Thus the angles where the polarization is equal to zero for singly scattered light are preserved for semi-infinite layers as well.

The dependence of the degree of polarization at fixed incidence angles as a function of the probability of photon absorption, calculated with Eq. (36) and the vector radiative transfer code, is presented in Fig. 3 at $\theta_0 = 30°, 37°, 0°$ for the same phase matrix as in Figs. 1 and 2. One can see that the accuracy of Eq. (36) is better than 2% at $\beta < 0.1$ at these geometries.

4. CONCLUSIONS

In conclusion, Eqs. (20) and (21) can be used for studies of the reflection and transmission matrices of optically thick disperse media, provided that the reflection matrix of a semi-infinite medium with the same microstructure is known. The phase matrix of the scattering medium under investigation can be arbitrary. However, the absorption of light by particles should be rather low, which is usually the case in the visible region of the electromagnetic spectrum.

The formulas obtained open new possibilities for the parameterization of global optical characteristics of disperse media. They also could be useful in the solution of inverse problems, particularly those that involve spectropolarimetric measurements.

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